above, we obtain equations for the determination of a and θ, which are easily integrated. The subsequent calculations are obvious.

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PURE SHEAR OF AN ELASTIC HALFSPACE WITH A SYSTEM OF CRACKS

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We consider a dynamic mixed problem for an elastic halfspace, weakened by a system of two-dimensional cracks and subject to conditions of anti-plane deformation.

We raise the problem of determining the jump in the stresses at the cracks in an elastic halfspace when shear displacements on the cracks are known. Using the method developed in [1, 2] we reduce the system of integral equations for the mixed problem to an equivalent system of linear algebraic equations with a completely continuous operator. We analyze the problems relating to the solvability of the integral equations and the infinite system. Investigation of the solution in the zero approximation is given.

The dynamics of an elastic halfspace with a crack was studied in [3, 4] wherein the main emphasis was focused on problems relating to crack propagation and the diffraction of elastic waves by the cracks.

1. We assume that the planes of the cracks J_n are of width l_n $(n = 1,2,\ldots, N)$, are directed parallel to the space coordinate z, and are unbounded on both sides (Fig. 1).

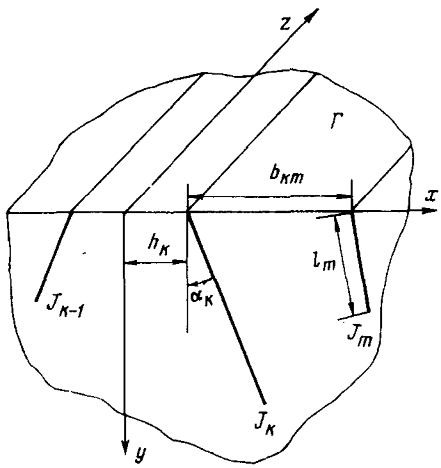

Fig. 1

On each plane of a crack tangential displacements are given, which vary harmonically with time t with circular frequency ω and are directed parallel to the z-axis:

$$W_n^*(s,\ t) = \operatorname{Re} W_n^{\pm}(s)\, e^{-i\omega t}, \qquad 0 \leqslant s \leqslant l_n$$

The local coordinate s along the edge of the crack is reckoned from the free boundary Γ of the elastic halfspace. Plus and minus signs refer, respectively, to the right and left edges of the crack J_n.

Under the conditions of stationary oscillations we are required to determine the jumps in the stresses at the cracks

$$[\tau_n^*(s,\ t)] = \operatorname{Re}[\tau_n(s)]\, e^{-i\omega t}, \qquad 0 \leqslant s \leqslant l_n$$

The problem so formulated reduces to the following boundary value problem for the Helmholtz equation:

$$\Delta w^\circ + k^2 w^\circ = 0, \qquad k^2 = D\omega^2\mu^{-1} \tag{1.1}$$

$$\left.\frac{\partial w^\circ}{\partial y}\right|_\Gamma = 0, \qquad w^\circ\,|_{J_n^{\pm}} = W_n^{\pm}(s)$$

$$0 \leqslant s \leqslant l_n$$

Here w° is the complex amplitude of the displacements in the elastic halfspace, D is the density of the material, and μ is the shear modulus.

We employ the Green's function method [5] to obtain the integral equation for the mixed problem. Imposing on the Green's function $G(\xi,\ \eta \mid x,\ y)$ for Eq. (1.1) the additional condition $(\partial G / \partial\eta)\,|_{\eta=0} = 0$, we arrive at a system of integral equations for the unknown jumps in the stresses at the cracks. In dimensionless form this system has the following form:

$$-\frac{1}{\pi}\sum_{n=1}^{N}\int_0^1 k_{mn}(r,\ \rho)\, q_n(\rho)\, d\rho = f_m(r) \tag{1.2}$$

$$2\mu q_n(r) = [\tau_n(r)]\, l_n l_m^{-1}, \qquad 0 \leqslant r \leqslant 1$$

$$l_n w_n^{\pm}(r) = W_n^{\pm}(s), \qquad l_n r = s$$

$$2f_m(r) = w_m^{+}(r) + w_m^{-}(r)$$

$$k_{mn}(r,\ \rho) = K_0(\varkappa_n R_{mn}^{+}) + K_0(\varkappa_n R_{mn}^{-})$$

$$R_{mn}^{\pm} = [(r - r_{mn}^{\pm})^2 + (\rho - \rho_{mn}^{\pm})^2 \pm 2(r - r_{mn}^{\pm})(\rho - \rho_{mn}^{\pm})\cos\psi_{mn}^{\pm}]^{1/2}$$

$$r_{mn}^{\pm} = d_{mn}\cos\alpha_n / \sin\psi_{nm}^{\pm}, \qquad d_{mn} l_n = b_{mn} = h_m - h_n$$

$$\rho_{mn}^{\pm} = d_{mn}\cos\alpha_m / \sin\psi_{nm}^{\pm}, \qquad \psi_{mn}^{\pm} = \alpha_n \pm \alpha_m$$

$$\varkappa_n = -i\lambda_n, \qquad \lambda_n = kl_n, \qquad m = 1, 2, \ldots, N$$

Here $K_0(z)$ is the MacDonald function of zero order.

We make the following assumptions relative to the dimensionless parameters introduced above:

1) The crack dimensions are sufficiently large in comparison with the length of the shear wave, i.e. $\lambda_n \gg 1$.

2) The mutual separation of the cracks on the free surface Γ is also sufficiently large in comparison with the crack dimensions, i.e. $d_{mn} \gg 1$.

3) $|r_{um}^{\pm}| \gg 1, \qquad |\rho_{mn}^{\pm}| \gg 1$.

One of the conditions sufficient for the satisfaction of (3) is the requirement

$$d \geqslant 2\operatorname{tg} \alpha^* \cos \alpha_* \tag{1.3}$$

$$\alpha_* = \min_n |\alpha_n|, \quad \alpha^* = \max_n |\alpha_n|, \quad d = \min_{m,n} d_{mn}$$

It follows from the condition (1.3), by virtue of the assumption (2), that

$$\alpha_0 \leqslant \alpha^* < \pi / 2, \qquad 1 < 2\alpha_0 < \pi \tag{1.4}$$

In the case $d \gg 1$ the quantity α_0 turns out to be close to $\pi / 2$ and the inequality (1.4) guarantees a sufficiently wide range of variation of the angles α_n, a situation which in some measure justifies the certain artificiality of condition (3).

2. Assuming that the right side of the integral equation (1.2) is representable by a Kontorovich-Lebedev integral, we restric our consideration to the case [1]

$$f_m(r) = I_\eta(\varkappa_m r) I_\eta^{-1}(\varkappa_m) \tag{2.1}$$

Upon taking into account the relations (3), the addition theorems for Bessel functions, and the Cramer formulas [6], we can transform the functions $k_{mn}(r, \rho)$, respectively, to the form (the first of the integrals is to be understood in the principal value sense)

$$k_{mm}(r, \rho) = \frac{1}{\pi i} \int_{-\infty}^{\infty} K_{-i\tau}(\varkappa_m \rho) I_{-i\tau}(\varkappa_m r) T_m(\tau)\, d\tau \tag{2.2}$$

$$k_{mn}(r, \rho) = \sum_{\nu=1}^{\infty} \varepsilon_\nu E_{\nu mn}(\varkappa_n \rho) I_\nu(\varkappa_n r)$$

$$E_{\nu mn}(u) = \frac{2}{\pi} \int_{-\infty}^{\infty} K_{-i\tau}(u) K_{-i\tau+\nu}(\beta_{mn}) \varphi_{mn}(\tau, i\nu)\, d\tau$$

$$T_m(\tau) = 2 \operatorname{ch} A_m\tau \operatorname{ch} B_m\tau / \operatorname{sh} \pi\tau$$

$$\varphi_{mn}(\zeta, z) = \begin{cases} \exp A_n z \operatorname{ch} A_m\zeta, & n \leqslant m - 1 \\ \exp B_m\zeta \operatorname{ch} B_n z, & n \geqslant m + 1 \end{cases}$$

$$A_n = \pi / 2 - \alpha_n, \qquad B_n = \pi / 2 + \alpha_n$$

$$\beta_{mn} = \varkappa_n \alpha_{mn}, \qquad \varepsilon_\nu = 1 + \operatorname{sgn}(\nu - 1)$$

The $I_p(z)$ and $K_p(z)$ here are modified Bessel functions. Using the method of residues, we can write the function $k_{mm}(r, \rho)$ in the form of the following series:

$$k_{mm}(r, \rho) = \sum_{\nu=1}^{\infty} s_\nu \begin{pmatrix} K_\nu(\varkappa_m r) I_\nu(\varkappa_m \rho), & \rho < r \\ I_\nu(\varkappa_m r) K_\nu(\varkappa_m \rho), & \rho > r \end{pmatrix}, \qquad s_\nu = {}^1/_2\, \varepsilon_\nu \operatorname{res} T_m(\tau)|_{\tau=-i\nu} \tag{2.3}$$

where the ζ are the poles of the function $T_m(\tau)$ in the upper halfplane. Let z_l be the zeros of this function in this same halfplane. Using the method given in [1, 2], we reduce the system of integral equations (1.2) with the right side (2.1) to an equivalent system of linear algebraic equations. For this we seek the unknown functions $q_n(r)$ in the form

$$rq_n(r) = x_0(n)\frac{I_\eta(\varkappa_n r)}{I_\eta(\varkappa_n)} + \sum_{l=1}^{\infty} x_l(n)\frac{I_{-iz_l}(\varkappa_n r)}{I_{-iz_l}(\varkappa_n)} \qquad (2.4)$$

where the x_l are suitably defined constants. We substitute the expressions (2.2)–(2.4) into the left side of equation (1.2) and carry out the integration. As a result of the integration we obtain series of Dirichlet type in the functions $I_\nu(\varkappa_m r)$ $(\nu = 1,2,\ldots)$. Upon requiring that these series yield the right side of Eq. (2.1) we arrive at the following system of matrix equations

$$A(m)X(m) + \sum_{n=1}^{N}{}' D(m, n)X(n) = B(m) \qquad (2.5)$$

$$A(m) = \{a_{rl}(m)\} = \frac{iW[I_{-iz_l}(\varkappa_m), K_{-i\zeta_r}(\varkappa_m)]}{(\zeta_r^2 - z_l^2) I_{-iz_l}(\varkappa_m) K_{-i\zeta_r}(\varkappa_m)}$$

$$D(m, n) = \{d_{rl}(m, n)\} = \frac{K_{i(z_l-\zeta_r)}(\beta_{mn})\varphi_{mn}(\zeta_r, z_l)}{\sigma_{rm} z_l I_{-iz_l}(\varkappa_n) K_{-i\zeta_r}(\varkappa_m)}$$

$$B(m) = \{b_r(m)\} = x_0(m)\frac{iW[I_\eta(\varkappa_m), K_{-i\zeta_r}(\varkappa_m)]}{(\zeta_r^2 + \eta^2) I_\eta(\varkappa_m) K_{-i\zeta_r}(\varkappa_m)} -$$

$$\sigma_{rm}^{-1}\sum_{n=1}^{N}{}' x_0(n)\frac{K_{\eta+i\zeta_r}(\beta_{mn})\varphi_{mn}(\zeta_r, \eta)}{\eta I_\eta(\varkappa_n) K_{-i\zeta_r}(\varkappa_m)}$$

$$X(n) = \{x_l(n)\}, \qquad W[x, y] = x'y - xy'$$

$$\sigma_{rm} = \pi^{-1}\varepsilon_r\varkappa_m(1 + \operatorname{ch} 2\alpha_m\zeta_r / \operatorname{ch}\pi\zeta_r)$$

$$x_0(n) = T_n^{-1}(i\eta), \qquad m = 1, 2, \ldots, N$$

where the prime on the summation sign means that the term for which $n = m$ is to be deleted. The equations (2.5) represent necessary and sufficient conditions for the solvability of Eq. (1.2) with the right side given by equation (2.1) in the class of solutions representable in the form (2.4) by virtue of the minimality of the system of functions $\{I_\nu(\varkappa_m r)\}$ $(\nu = 1,2,\ldots)$ on the interval $0 \leqslant r \leqslant 1$ (see [1]). The latter property shows the equivalence of the infinite system to the integral equation in the sense that the infinite system and the integral equation are both solvable (or both nonsolvable) and they automatically have the same number of solutions.

3. Using the uniform asymptotic estimates for the behavior of the modified Bessel functions, we can readily establish that when $\lambda_n \gg 1$ and $d_{mn} \gg 1$ the elements of the matrices $A(m)$ are sufficiently close to the elements of the matrix $A = \{(\zeta_r - z_l)^{-1}\}$, and the elements of the matrices $D(m, n)$ are small of the order

$$O[\exp(-|\zeta_r - z_l|\delta_{mn})], \qquad \delta_{mn} = \ln d_{mn}$$

An infinite system with a matrix whose principal part coincides with the matrix $A(m)$ was considered in [1], where it was shown that an equation of the type (2.5) can be reduced to an equivalent equation of the second kind

$$X(m) = A^{-1}[A - A(m)]X(m) + A^{-1}B(m) - \sum_{n=1}^{N}{}' A^{-1}D(m, n)X(n), \quad m = 1, 2, \ldots, N \tag{3.1}$$

We note that in the given case it is necessary in the process of determining the matrix A^{-1} to factor the function $T_m(z)z^{-1}$ (which has a double pole on the real axis) with respect to the real axis in the complex z plane.

It is not difficult to verify that the matrices on the right side of the system (3.1) generate completely continuous operators in the space of the sequences $s(\sigma)$, $0 < \sigma < {}^1/_2$, with the metric

$$\|X\|_{s(\sigma)} = \sup_l |x_l l^\sigma| < \infty, \qquad \lim_l |x_l l^\sigma| = 0 \qquad (l \to \infty)$$

The complete continuity of the operators of the system (3.1) enables us to make known inferences from this theory relative to the solvability of the infinite system itself and the integral equation (1.2) corresponding to it. In the case $d \gg 1$, $\lambda \gg 1$ ($\lambda = \min_n \lambda_n$) the operators on the right side of the system (3.1) are contractive and the infinite system is then uniquely solvable; the unique solution can be found by the method of successive approximations.

4. We study the solution of the problem in the zero approximation. As the solution of the system (3.1) in the zero approximation we choose the matrices

$$X^\circ(m) = A^{-1}B(m), \qquad m = 1, 2, \ldots, N$$

since the remaining terms of the system are negligibly small in the norm of $s(\sigma)$, $0 < \sigma < {}^1/_2$. Taking the expressions for $x_l^\circ(m)$ to the right side of Eq. (2.4) and summing the resulting series with the aid of contour integration, we arrive finally at the following asymptotic expression for the unknown functions ($T_+(z, m)$ resulting from the factorization of $T_m(z)\ z^{-1}$):

$$q_m(r) \sim (\ln r)^{-1/2} v_m(r, \eta)[1 + O(\sqrt{\ln r})], \qquad r \to 1 \tag{4.1}$$

$$v_m(r, \eta) = \frac{1}{2\eta\sqrt{\pi}}\left[\frac{V(\varkappa_m) + \eta}{T_+(i\eta, m)} - \frac{V(\varkappa_m) - \eta}{T_+(-i\eta, m)} + \Delta(\eta, m)\right][1 - 2\theta(\alpha_m)\ln r]$$

$$\Delta(\eta, m) = i\sqrt{\pi}[\varkappa_m I_\eta(\varkappa_m)]^{-1} \sum_{n=1}^{N}{}' \varphi_{mn}(0, -\eta)x_0(n)d_{mn}^{-\eta}$$

$$V(\varkappa) = I_\eta'(\varkappa)I_\eta^{-1}(\varkappa), \qquad \theta(\alpha) = {}^1/_{12} + [{}^1/_4 - (\alpha/\pi)^2]^{-1} \tag{4.2}$$

$$q_m(r) \sim Q(\eta, m)(kr/2)^{\eta-1}[1 + o(r)] \qquad r \to 0$$

$$0 < \operatorname{Re}\eta < 1, \qquad m = 1, 2, \ldots, N$$

The expressions (4.1) and (4.2) show, subject to the restrictions $d \gg 1$ and $\lambda \gg 1$, that the inclination of the cracks to the free boundary of the halfspace has practically no influence on the distribution of the contact stresses close to the ends and that the influence on the m-th crack of all the remaining cracks is very insignificant and is characterized by the quantity

$$\Delta(\eta, m) = O(\lambda^{-1/2} d^{-\eta})$$

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NOTE ON THE PROOF OF CONVERGENCE OF THE METHOD OF FINITE ELEMENTS

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A simplification and generalization of the proof presented in [1] is given.

1. Let V be the region occupied by an elastic body subjected to deformation, and V^e be subregions representing finite elements. For simplicity we set, as in [1], $UV^e = V$. The field of displacements $\mathbf{f}^e$ in each subregion is approximated by formula [2]

$$\mathbf{f}^e = \mathbf{N}^e \delta^e \qquad (1.1)$$

where $\mathbf{f}^e$ is the displacement vector of points within the element of number e, δ^e is the vector of nodal displacements, and $\mathbf{N}^e$ is a rectangular matrix whose elements are functions of coordinates. The approximation of the displacement throughout region V can be defined by

$$\mathbf{f}_n = \sum_s \sum_{k=1}^{n} \mathbf{f}_{kn}^s \delta_{ks} \qquad (1.2)$$

where δ_{ks} are components of the displacement vector of the k-th node and functions $\mathbf{f}_{kn}{}^s$ are piecewise determinate and nonzero only in elements one of whose vertices bears the number k. The system of equations of the method of finite elements is derived by minimizing the functional of energy over the set of functions of the form (1.2). A similar method of solving the problem of minimization of the energy functional was used in [3, 4]. The sequence of approximate solutions converges to the exact (generalized) one, if conditions (1)–(3) of the convergence theorem are satisfied (see Sect. 19 of [3]).

2. Similar results were obtained in [5] for the case when the operator of the boundary value problem contains derivatives of an arbitrary order. Let us briefly consider the con-